INSTITUT PENYELIDIKAN MATEMATIK

OLIMPIAD MATEMATIK UNIVERSITI MALAYSIA 2024
OMUM2024

PEPERIKSAAN PERINGKAT AKHIR

Tarikh   :   08 Jun 2024

Masa   :   9.45 pagi – 12.45 tengah hari

Tempoh   :   3 jam

Arahan kepada calon:

1. Jawab **SEMUA** soalan.

2. Kalkulator adalah **TIDAK** dibenarkan sepanjang peperiksaan berlangsung.

3. Soalan adalah dalam bahasa Inggeris.

4. Markah diberi untuk jalan kerja dan jawapan yang tepat.

Question 1

Solve for complex number $z = x + iy$ if $|z + 2| = 1 + i\overline{z}$ where $|z|$ and $\overline{z}$ denote the modulus of z and the complex conjugate of z, respectively. Show that there is only one solution.

[10 marks]

Question 2

Given a nondegenerate triangle with integer sidelengths. It is known that one angle of the triangle is $\frac{2\pi}{3}$ and one of the sides has length equal to the average of the other two. What is the smallest possible parameter of this triangle?

[10 marks]

Question 3.

Calculate the probability of randomly chosen numbers α and β in the closed interval $[0, 1]$ such that $\alpha + \beta \leq 1$ and their product does not exceed $\frac{8}{81}$.

[10 marks]

Question 4.

Show that there is a bijection $f : \{1, 2, 3, \dots\dots, 2024\} \to \{1, 2, 3, \dots\dots, 2024\}$ such that for any $1 \leq i < j < k \leq 2024$,

$$f(i) + f(k) \neq 2f(j).$$

[10 marks]

Question 5

Let $\mathcal{A}$ be a collection of distinct subsets of $\{1, 2, 3, \ldots, 2024\}$ with $|\mathcal{A}| \geq 2^{1013}$, where $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. Prove that there must be three distinct sets $A, B, C,$ in $\mathcal{A}$ such that

$$B \cap C \subseteq A \subseteq B \cup C.$$

[10 marks]

Question 6

For any positive integer n and real number x, let $A_{n,x}$ be an $n \times n$ matrix given by

$$A_{n,x} = \begin{pmatrix} 1 & x & \cdots & x^{n-1} & x^n \\ x & 1 & x & \cdots & x^{n-1} \\ \vdots & x & \ddots & \ddots & \vdots \\ x^{n-1} & \vdots & \ddots & 1 & x \\ x^n & x^{n-1} & \cdots & x & 1 \end{pmatrix}.$$

Determine if $A_{n,x}$ is invertible on the real number line $\mathbb{R}$ and find its inverse if exist.

[10 marks]